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On the existence of certain error formulas for a special class of ideal projectors

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Abstract

In this paper, we focus on a special class of ideal projectors. With the aid of algebraic geometry, we prove that for this special class of ideal projectors, there exist "good" error formulas as defined by C. de Boor. Furthermore, we completely analyze the properties of the interpolation conditions matched by this special class of ideal projectors, and show that the ranges of this special class of ideal projectors are the minimal degree interpolation spaces with regard to their associated interpolation conditions.

Keywords: Ideal projectors; Error formulas; Reduced Gröbner bases; Minimal degree interpolation spaces

1. Introduction

The problem of polynomial interpolation is to construct a function p, belonging to a finite-dimensional subspace of $\mathbb{F}[\boldsymbol{x}]$, that agrees with another given function f on a set of interpolation conditions, where $\mathbb{F}[\boldsymbol{x}] := \mathbb{F}[x_1, \ldots, x_d]$ denotes the polynomial ring in d variables over the field $\mathbb{F}$. If there exists a unique solution of the interpolation problem for every f, we say that the interpolation problem is poised. It is important to make the comment that $\mathbb{F}$ is a field of characteristic zero in this paper, for example $\mathbb{F} = \mathbb{Q}, \mathbb{R}, \mathbb{C}$.

Error formulas for polynomial interpolation give explicit representations for the interpolation error. de Boor [5] derived a formula for the interpolation error. In terms of *Newton*

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fundamental polynomials, Sauer and Xu [11,12] presented Sauer–Xu error formulas for polynomial interpolation, whose interpolation conditions have certain constraints. Afterward, de Boor [6] discussed the error formulas in tensor-product and Chung–Yao interpolation. In 1998, Waldron [18] investigated the error in linear interpolation at the vertices of a simplex.

As an elegant form of multivariate approximate, ideal interpolation provides a natural link between polynomial interpolation and algebraic geometry. According to G. Birkhoff's definition [2], a linear idempotent operator P on $\mathbb{F}[\boldsymbol{x}]$ is called an *ideal projector* if $\ker P$ is an ideal. In the theory of ideal interpolation, we are interested in finite-rank ideal projectors. A *finite-rank ideal projector* refers to the ideal projector whose range is a finite-dimensional subspace of $\mathbb{F}[\boldsymbol{x}]$. As mentioned by Carl de Boor, one reason for choosing ideal interpolation in the first place is the resulting possibility of writing the error formulas as in the following definition.

Definition 1 (*[7]*)**.** Let P be an ideal projector and $\{h_1, \ldots, h_m\}$ be an ideal basis for $\ker P$. We say that the basis $\{h_1, \ldots, h_m\}$ admits a "good" error formula if there exist homogeneous polynomials H_j and linear operators $C_j : \mathbb{F}[\boldsymbol{x}] \to \mathbb{F}[\boldsymbol{x}]$, $j = 1, \ldots, m$ such that for all $f \in \mathbb{F}[\boldsymbol{x}]$,

$$H_j(D)h_k = \delta_{j,k} \quad \text{for all } j, k = 1, \ldots, m$$

and

$$f - Pf = \sum_{j=1}^{m} C_j(H_j(D)f)h_j,$$

where $H_j(D)$ will be defined in Section 2.

We say that P has a "good" error formula if there exists an ideal basis $\{h_1, \ldots, h_m\}$ for $\ker P$ that admits a "good" error formula.

It is no surprise that every ideal projector in univariate polynomial ring has a "good" error formula [13,16]. When we turn to multivariate interpolation, things change greatly. de Boor [6] proved the existence of "good" error formulas for tensor-product and Chung–Yao interpolation. However, Shekhtman [17] showed that for a specific form of ideal interpolation by linear polynomials in two variables, such a "good" error formula does not exist. Hence, the study of the type of ideal projectors with "good" error formulas is a rather complicated topic.

In this paper, we deal with a special class of ideal projectors, and prove the existence of "good" error formulas for this class of ideal projectors. It should be noted that the construction of the linear operators C_j, $1 \le j \le m$, as in Definition 1 will be the subject of our future work. Moreover, we discuss the properties of the interpolation conditions matched by this special class of ideal projectors. The main results of this paper will be put in Section 3. The next section, Section 2, is devoted as a preparation for this paper.

2. Preliminaries

In this section, we will introduce some notation and recall some basic facts about ideal interpolation and algebraic geometry. For more details, we refer the reader to [7,16,4,1].

Throughout the paper, we use $\mathbb{N}$ to stand for the set of nonnegative integers, and use boldface letters to express tuples and denote their entries by the same letter with subscripts, for example, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$. For arbitrary $\boldsymbol{\alpha} \in \mathbb{N}^d$, we define $\boldsymbol{\alpha}! = \alpha_1! \ldots \alpha_d!$.

A monomial $\boldsymbol{x}^{\boldsymbol{\alpha}}$ is a power product of the form $x_1^{\alpha_1} \dots x_d^{\alpha_d}$ with $\boldsymbol{\alpha} \in \mathbb{N}^d$. We denote by $\mathbb{T}(\boldsymbol{x}) := \mathbb{T}(x_1, \dots, x_d)$, the set of all monomials in $\mathbb{F}[\boldsymbol{x}]$. For a polynomial $f(\boldsymbol{x}) = \sum_{\boldsymbol{\alpha} \in \mathbb{N}^d} c_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}} \in \mathbb{F}[\boldsymbol{x}]$ with $0 \neq c_{\boldsymbol{\alpha}} \in \mathbb{F}$, we write the associated differential operator for f in the form

$$f(D) = \sum_{\boldsymbol{\alpha} \in \mathbb{N}^d} c_{\boldsymbol{\alpha}} D^{\boldsymbol{\alpha}},$$

where

$$D^{\boldsymbol{\alpha}} = \frac{\partial^{\alpha_1 + \cdots + \alpha_d}}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}}.$$

Henceforward, we use $\leq$ to denote the usual product order on $\mathbb{N}^d$. For $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^d$, $\boldsymbol{\alpha} \leq \boldsymbol{\beta}$ if and only if $\alpha_i \leq \beta_i$, $i = 1, \dots, d$. In particular, $\boldsymbol{\alpha} < \boldsymbol{\beta}$ if and only if $\boldsymbol{\alpha} \leq \boldsymbol{\beta}$ and $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$. A finite subset $\mathcal{A} \subset \mathbb{N}^d$ is *lower* if for every $\boldsymbol{\alpha} \in \mathcal{A}$, $\boldsymbol{0} \leq \boldsymbol{\beta} \leq \boldsymbol{\alpha}$ implies $\boldsymbol{\beta} \in \mathcal{A}$.

A finite monomial set $\mathcal{O} \subset \mathbb{T}(\boldsymbol{x})$ is called an *order ideal* if it is closed under monomial division, namely $\boldsymbol{t} \in \mathcal{O}$ and $\boldsymbol{t}' | \boldsymbol{t}$ imply $\boldsymbol{t}' \in \mathcal{O}$. For an order ideal $\mathcal{O} \subset \mathbb{T}(\boldsymbol{x})$, the *corner set* of $\mathcal{O}$, denoted by $\mathcal{C}[\mathcal{O}]$, is the set

$$\mathcal{C}[\mathcal{O}] = \{\boldsymbol{t} \in \mathbb{T}(\boldsymbol{x}) : \boldsymbol{t} \notin \mathcal{O}, x_i | \boldsymbol{t} \Rightarrow \boldsymbol{t}/x_i \in \mathcal{O},\ 1 \leq i \leq d\}.$$

Fix a monomial order $\prec$ on $\mathbb{T}(\boldsymbol{x})$, for all $0 \neq f \in \mathbb{F}[\boldsymbol{x}]$, we may write

$$f = c_{\boldsymbol{\gamma}^{(1)}} \boldsymbol{x}^{\boldsymbol{\gamma}^{(1)}} + c_{\boldsymbol{\gamma}^{(2)}} \boldsymbol{x}^{\boldsymbol{\gamma}^{(2)}} + \cdots + c_{\boldsymbol{\gamma}^{(r)}} \boldsymbol{x}^{\boldsymbol{\gamma}^{(r)}}$$

where $0 \neq c_{\boldsymbol{\gamma}^{(i)}} \in \mathbb{F}$, $\boldsymbol{\gamma}^{(i)} \in \mathbb{N}^d$, $i = 1, \dots, r$, and $\boldsymbol{x}^{\boldsymbol{\gamma}^{(1)}} \succ \boldsymbol{x}^{\boldsymbol{\gamma}^{(2)}} \succ \cdots \succ \boldsymbol{x}^{\boldsymbol{\gamma}^{(r)}}$. We shall call $\mathrm{LT}_{\prec}(f) := c_{\boldsymbol{\gamma}^{(1)}} \boldsymbol{x}^{\boldsymbol{\gamma}^{(1)}}$ the *leading term* and $\mathrm{LM}_{\prec}(f) := \boldsymbol{x}^{\boldsymbol{\gamma}^{(1)}}$ the *leading monomial* of f.

Given an ideal $\mathcal{I}$ and a monomial order $\prec$, there exists a unique reduced Gröbner basis $G_{\prec}$ for $\mathcal{I}$ w.r.t. $\prec$. Suppose that $G_{\prec} = \{g_1, \dots, g_m\}$, then the set

$$\mathcal{N}_{\prec}(\mathcal{I}) := \{\boldsymbol{x}^{\boldsymbol{\alpha}} \in \mathbb{T}(\boldsymbol{x}) : \mathrm{LT}_{\prec}(g_j) \nmid \boldsymbol{x}^{\boldsymbol{\alpha}},\ \text{for all } 1 \leq j \leq m\}$$

is called the *Gröbner éscalier* of $\mathcal{I}$ w.r.t. $\prec$. From the theory of Gröbner bases, we know that $\mathcal{N}_{\prec}(\mathcal{I})$ is an order ideal, and

$$\mathcal{C}[\mathcal{N}_{\prec}(\mathcal{I})] = \{\mathrm{LT}_{\prec}(g_1), \dots, \mathrm{LT}_{\prec}(g_m)\}.$$

If P is a finite-rank ideal projector on $\mathbb{F}[\boldsymbol{x}]$, then there are two important subsets of $\mathbb{F}[\boldsymbol{x}]$ associated with P. The range of P is defined as

$$V := \mathrm{ran} P = \{p \in \mathbb{F}[\boldsymbol{x}] : p = Pf \text{ for some } f \in \mathbb{F}[\boldsymbol{x}]\},$$

which is a finite-dimensional subspace of $\mathbb{F}[\boldsymbol{x}]$, and the kernel space of P

$$\ker P = \{g \in \mathbb{F}[\boldsymbol{x}] : Pg = 0\},$$

which forms a zero-dimensional ideal in $\mathbb{F}[\boldsymbol{x}]$. Furthermore, as an infinite-dimensional $\mathbb{F}$-vector space, $\mathbb{F}[\boldsymbol{x}]$ has a corresponding dual space $(\mathbb{F}[\boldsymbol{x}])'$. An ideal projector P on $\mathbb{F}[\boldsymbol{x}]$ also has a dual projector P^* on $(\mathbb{F}[\boldsymbol{x}])'$, and the range of P^* is

$$\Lambda := \mathrm{ran} P^* = (\ker P)^{\perp} = \{\lambda \in (\mathbb{F}[\boldsymbol{x}])' : \ker P \subset \ker \lambda\}.$$

Indeed, Λ is the set of interpolation conditions matched by P. It is easy to see that $\dim \Lambda = \dim V$ and

$$\ker\Lambda := \{f \in \mathbb{F}[\boldsymbol{x}] : \lambda(f) = 0 \text{ for all } \lambda \in \Lambda\} = \ker P$$

which satisfies

$$\ker\Lambda \cap V = \{0\}.$$

The following theorems summarize some of the simple properties of ideal projectors.

Theorem 1 ([7]). *A linear operator $P : \mathbb{F}[\boldsymbol{x}] \to \mathbb{F}[\boldsymbol{x}]$ is an ideal projector if and only if the equality*

$$P(fg) = P(fPg)$$

holds for all $f, g \in \mathbb{F}[\boldsymbol{x}]$.

Theorem 2 ([14,15]). *A linear operator $P : \mathbb{F}[\boldsymbol{x}] \to \mathbb{F}[\boldsymbol{x}]$ is an ideal projector if and only if the operator $P' := I - P$ satisfies*

$$P'(fg) = fP'(g) + P'(fPg)$$

for all $f, g \in \mathbb{F}[\boldsymbol{x}]$.

3. Main results

In this section, we will describe a special class of ideal projectors with "good" error formulas in terms of ideal bases and interpolation conditions respectively.

3.1. Representation in terms of ideal bases

Following Sauer [10], we refer to a reduced Gröbner basis G for an ideal $\mathcal{I}$ as a *universal Gröbner basis* if G is a unique reduced Gröbner basis for $\mathcal{I}$, independent of the monomial order. Now, we begin with an easy lemma about the universal Gröbner bases.

Lemma 3. *If the ideal* $\ker P$ *has a reduced Gröbner basis $G = \{g_1, \ldots, g_m\}$ w.r.t. some monomial order, and the polynomials of G have the form*

$$g_j = \boldsymbol{x}^{\boldsymbol{\alpha}^{(j)}} - \sum_{0 \le \boldsymbol{\beta} < \boldsymbol{\alpha}^{(j)}} c_{j,\boldsymbol{\beta}} \boldsymbol{x}^{\boldsymbol{\beta}}, \quad \text{for } 1 \le j \le m \text{ and } c_{j,\boldsymbol{\beta}} \in \mathbb{F}, \tag{1}$$

then G is a universal reduced Gröbner basis for $\ker P$ *w.r.t. any monomial order, and the monomial set*

$$\mathcal{O} = \{\boldsymbol{x}^{\boldsymbol{\beta}} \in \mathbb{T}(\boldsymbol{x}) : \boldsymbol{x}^{\boldsymbol{\alpha}^{(j)}} \nmid \boldsymbol{x}^{\boldsymbol{\beta}}, \text{ for all } 1 \le j \le m\} \tag{2}$$

is the unique Gröbner éscalier of $\ker P$ *w.r.t. any monomial order.*

Proof. For an arbitrary j with $1 \le j \le m$ and an arbitrary $\boldsymbol{\beta} \in \mathbb{N}^d$ with $0 \le \boldsymbol{\beta} < \boldsymbol{\alpha}^{(j)}$, we have that $\boldsymbol{x}^{\boldsymbol{\beta}} \mid \boldsymbol{x}^{\boldsymbol{\alpha}^{(j)}}$. Suppose that $\prec$ is an arbitrary monomial order, then $\boldsymbol{x}^{\boldsymbol{\beta}} \mid \boldsymbol{x}^{\boldsymbol{\alpha}^{(j)}}$ together with

$\boldsymbol{\beta} \neq \boldsymbol{\alpha}^{(j)}$ implies $\boldsymbol{x}^{\boldsymbol{\beta}} \prec \boldsymbol{x}^{\boldsymbol{\alpha}^{(j)}}$. Consequently, for an arbitrary monomial order $\prec$, $\mathrm{LT}_{\prec}(g_j) = \boldsymbol{x}^{\boldsymbol{\alpha}^{(j)}}$ with $1 \leq j \leq m$, and S-polynomial of g_i and g_j with $1 \leq i < j \leq m$ is the combination

$$S(g_i, g_j) = \frac{\mathrm{LCM}(\boldsymbol{x}^{\boldsymbol{\alpha}^{(i)}}, \boldsymbol{x}^{\boldsymbol{\alpha}^{(j)}})}{\boldsymbol{x}^{\boldsymbol{\alpha}^{(i)}}} g_i - \frac{\mathrm{LCM}(\boldsymbol{x}^{\boldsymbol{\alpha}^{(i)}}, \boldsymbol{x}^{\boldsymbol{\alpha}^{(j)}})}{\boldsymbol{x}^{\boldsymbol{\alpha}^{(j)}}} g_j,$$

where $\mathrm{LCM}(\boldsymbol{x}^{\boldsymbol{\alpha}^{(i)}}, \boldsymbol{x}^{\boldsymbol{\alpha}^{(j)}})$ is the least common multiple of $\boldsymbol{x}^{\boldsymbol{\alpha}^{(i)}}$ and $\boldsymbol{x}^{\boldsymbol{\alpha}^{(j)}}$.

Since G is a reduced Gröbner basis for $\ker P$ w.r.t. some monomial order, it follows that $S(g_i, g_j)$ reduces to zero module G w.r.t. this monomial order. Indeed, for arbitrary monomial order $\prec$, $\mathrm{LT}_{\prec}(g_j) = \boldsymbol{x}^{\boldsymbol{\alpha}^{(j)}}$, it implies that $S(g_i, g_j)$ reduces to zero module G w.r.t. any monomial order. Therefore, we can say that G is a universal reduced Gröbner basis for $\ker P$ w.r.t. any monomial order. Furthermore, it follows that $\mathcal{O}$ is the unique Gröbner éscalier of $\ker P$ w.r.t. any monomial order. $\square$

Proposition 4. *Let $\{g_1, \ldots, g_m\}$ be a reduced Gröbner basis for* $\ker P$ *satisfying condition (1), and $\mathcal{O}$ be a monomial set as in (2). Then for every monomial $\boldsymbol{x}^{\boldsymbol{\gamma}} \in \mathbb{T}(\boldsymbol{x})$, there exist polynomials $A_{\boldsymbol{\gamma},j}$, $1 \leq j \leq m$ such that*

$$P'(\boldsymbol{x}^{\boldsymbol{\gamma}}) = \boldsymbol{x}^{\boldsymbol{\gamma}} - P(\boldsymbol{x}^{\boldsymbol{\gamma}}) = \sum_{j=1}^{m} A_{\boldsymbol{\gamma},j} g_j \tag{3}$$

and

$$A_{\boldsymbol{\gamma},j} = 0 \quad \textit{if } \boldsymbol{x}^{\boldsymbol{\alpha}^{(j)}} \nmid \boldsymbol{x}^{\boldsymbol{\gamma}}. \tag{4}$$

In other words,

$$P'(\boldsymbol{x}^{\boldsymbol{\gamma}}) = \boldsymbol{x}^{\boldsymbol{\gamma}} - P(\boldsymbol{x}^{\boldsymbol{\gamma}}) = \sum_{\boldsymbol{\alpha}^{(j)} \leq \boldsymbol{\gamma}} A_{\boldsymbol{\gamma},j} g_j. \tag{5}$$

Proof. For every $\boldsymbol{\gamma} \in \mathbb{N}^d$, define an ideal

$$J_{\boldsymbol{\gamma}} = \langle g_j : \boldsymbol{\alpha}^{(j)} \leq \boldsymbol{\gamma} \rangle.$$

To prove this proposition, it suffices to show that $\boldsymbol{x}^{\boldsymbol{\gamma}} - P(\boldsymbol{x}^{\boldsymbol{\gamma}}) \in J_{\boldsymbol{\gamma}}$ for every $\boldsymbol{x}^{\boldsymbol{\gamma}} \in \mathbb{T}(\boldsymbol{x})$. Assume not and let $\boldsymbol{x}^{\boldsymbol{\gamma}}$ be a monomial of least total degree such that $\boldsymbol{x}^{\boldsymbol{\gamma}} - P(\boldsymbol{x}^{\boldsymbol{\gamma}}) \notin J_{\boldsymbol{\gamma}}$. Since for every $\boldsymbol{x}^{\boldsymbol{\beta}} \in \mathcal{O}$,

$$0 = \boldsymbol{x}^{\boldsymbol{\beta}} - P(\boldsymbol{x}^{\boldsymbol{\beta}}) \in J_{\boldsymbol{\gamma}},$$

we know that $\boldsymbol{x}^{\boldsymbol{\gamma}} \notin \mathcal{O}$. Therefore, we can find some $1 \leq j \leq m$ such that $\boldsymbol{\alpha}^{(j)} \leq \boldsymbol{\gamma}$. Let $\boldsymbol{\delta} = \boldsymbol{\gamma} - \boldsymbol{\alpha}^{(j)} \geq \mathbf{0}$. By Lemma 3, we have

$$\boldsymbol{x}^{\boldsymbol{\delta}} g_j = \boldsymbol{x}^{\boldsymbol{\gamma}} - \sum_{\boldsymbol{\beta} \in \mathcal{O}, \boldsymbol{\beta} < \boldsymbol{\alpha}^{(j)}} c_{j,\boldsymbol{\beta}} \boldsymbol{x}^{\boldsymbol{\beta}+\boldsymbol{\delta}}.$$

Consequently,

$$\boldsymbol{x}^{\boldsymbol{\delta}} g_j = P'(\boldsymbol{x}^{\boldsymbol{\delta}} g_j) = P'(\boldsymbol{x}^{\boldsymbol{\gamma}}) - \sum_{\boldsymbol{\beta} \in \mathcal{O}, \boldsymbol{\beta} < \boldsymbol{\alpha}^{(j)}} c_{j,\boldsymbol{\beta}} P'(\boldsymbol{x}^{\boldsymbol{\beta}+\boldsymbol{\delta}}) \in J_{\boldsymbol{\gamma}}.$$

But for every $\boldsymbol{\beta}$ such that $\boldsymbol{\beta} \in \mathcal{O}$, $\boldsymbol{\beta} < \boldsymbol{\alpha}^{(j)}$ we have $\boldsymbol{\beta} + \boldsymbol{\delta} < \boldsymbol{\gamma}$. Recall that $\boldsymbol{x}^{\boldsymbol{\gamma}}$ is a monomial of least total degree such that $P'(\boldsymbol{x}^{\boldsymbol{\gamma}}) \notin J_{\boldsymbol{\gamma}}$. Hence, $P'(\boldsymbol{x}^{\boldsymbol{\beta}+\boldsymbol{\delta}}) \in J_{\boldsymbol{\beta}+\boldsymbol{\delta}} \subset J_{\boldsymbol{\gamma}}$. Since $J_{\boldsymbol{\gamma}}$ is an ideal, then $P'(\boldsymbol{x}^{\boldsymbol{\gamma}}) \in J_{\boldsymbol{\gamma}}$. This is a contradiction to our hypothesis. $\square$

We need a standard key lemma for factorization of homomorphisms.

Lemma 5 ([13,14]). *Let $A : X \to Y$ and $B : X \to Z$ be two linear operators between linear spaces X, Y and Z. Then there exists linear operator C such that*

$$A = CB$$

if and only if

$$\ker B \subset \ker A.$$

The fact that an ideal projector P has a "good" error formula depends on not only the ideal basis for $\ker P$, but also the choice of $\operatorname{ran} P$. Next is the main theorem of this paper, which states that the ideal projectors satisfying the conditions of Theorem 6 have "good" error formulas.

Theorem 6. *Suppose that an ideal* $\ker P$ *has a universal reduced Gröbner basis G satisfying the conditions of Lemma 3, and* $\operatorname{ran} P$ *is*

$$V = \operatorname{span}_{\mathbb{F}}\{\boldsymbol{x}^{\boldsymbol{\beta}} \in \mathbb{T}(\boldsymbol{x}) : \boldsymbol{x}^{\boldsymbol{\alpha}^{(j)}} \nmid \boldsymbol{x}^{\boldsymbol{\beta}},\ \textit{for all}\ 1 \leq j \leq m\}. \tag{6}$$

Then G is the ideal basis for $\ker P$ *that admits a "good" error formula.*

Proof. Define operators A_j on $\mathbb{T}(\boldsymbol{x})$ by letting

$$A_j(\boldsymbol{x}^{\boldsymbol{\gamma}}) = \begin{cases} A_{j,\boldsymbol{\gamma}}, & \text{if } \boldsymbol{\alpha}^{(j)} \leq \boldsymbol{\gamma}; \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

where $A_{j,\boldsymbol{\gamma}}$ are defined in (5) and extend $A_{j,\boldsymbol{\gamma}}$ by linearity on $\mathbb{F}[\boldsymbol{x}]$. Then by (5) and linearity, we have

$$f - Pf = P'f = A_j(f)g_j.$$

By (7),

$$\ker A_j \supseteq \operatorname{span}\{\boldsymbol{x}^{\boldsymbol{\gamma}} : \boldsymbol{x}^{\boldsymbol{\alpha}^{(j)}} \nmid \boldsymbol{x}^{\boldsymbol{\gamma}}\} = \ker\left(\frac{1}{\boldsymbol{\alpha}^{(j)}!} D^{\boldsymbol{\alpha}^{(j)}}\right).$$

Hence, by Lemma 5, there exist operators C_j such that $A_j = C_j \circ H_j(D)$ where $H_j(D) := \frac{1}{\boldsymbol{\alpha}^{(j)}!} D^{\boldsymbol{\alpha}^{(j)}}$. It is trivial to check that $H_j(D)(g_k) = \delta_{j,k}$. □

In the following, we will present some examples to illustrate the conclusion of Theorem 6.

Example 1. Let P be a Lagrange projector onto $\operatorname{span}_{\mathbb{F}}\{1, x_1, x_2, x_2^2\}$ with the interpolation point set $\{(1, 0), (1, 1), (1, 2), (2, 0)\} \subset \mathbb{F}^2$. Then by Theorem 6,

$$\{(x_1 - 1)(x_1 - 2), x_2(x_2 - 1)(x_2 - 2), x_2(x_1 - 1)\}$$

is the ideal basis for $\ker P$ that admits a "good" error formula. □

Example 2. Let P be an ideal projector onto $\operatorname{span}_{\mathbb{F}}\{1, x_1, x_2, x_1^2, x_1x_2, x_2^2, x_1^3\}$ given by

$$Px_1^2x_2 = 0$$
$$Px_2^3 = x_2$$
$$Px_1x_2^2 = x_1x_2$$
$$Px_1^4 = 2x_1^3 - x_1^2.$$

Then the ideal basis

$$\{x_1^2x_2 - Px_1^2x_2, x_2^3 - Px_2^3, x_1x_2^2 - Px_1x_2^2, x_1^4 - Px_1^4\}$$

admits a "good" error formula. □

Example 3. Let P be a Lagrange projector onto the $\mathrm{span}_{\mathbb{F}}\{1, x_1, x_2, x_3\}$ with the interpolation point set $\{(0, 0, 0), (0, 1, 0), (0, 0, 1), (1, 0, 1)\} \subset \mathbb{F}^3$. Then

$$\{x_1x_2, x_2x_3, x_1^2 - x_1, x_2^2 - x_2, x_3^2 - x_3, x_1x_3 - x_1\}$$

is the ideal basis for $\ker P$ that admits a "good" error formula. □

We select test functions

$$f_1(x_1, x_2) = (1 - x_1)^2 + (1 - x_2)^2 + 1,$$
$$f_2(x_1, x_2) = x_1^3 + x_2^3,$$
$$f_3(x_1, x_2, x_3) = (1 - x_1)^2 + (1 - x_2)^2 + (1 - x_3)^2 + 1,$$
$$f_4(x_1, x_2, x_3) = x_1^3 + x_2^3 + x_3^3$$

to illustrate the "good" error formulas about the ideal projectors in the above examples.

For Example 1, we have

$$f_1 - Pf_1 = (x_1 - 1)(x_1 - 2),$$
$$f_2 - Pf_2 = (x_1 + 3)(x_1 - 1)(x_1 - 2) + x_2(x_2 - 1)(x_2 - 2).$$

For Example 2, we get

$$f_1 - Pf_1 = 0,$$
$$f_2 - Pf_2 = x_2^3 - x_2.$$

For Example 3,

$$f_3 - Pf_3 = x_1^2 - x_1 + x_2^2 - x_2 + x_3^2 - x_3,$$
$$f_4 - Pf_4 = (x_1 + 1)(x_1^2 - x_1) + (x_2 + 1)(x_2^2 - x_2) + (x_3 + 1)(x_3^2 - x_3).$$

3.2. Representation in terms of interpolation conditions

Next, we will describe the properties of the interpolation conditions matched by the ideal projectors satisfying the conditions of Theorem 6.

Proposition 7. *Let P be an ideal projector, and $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ the set of interpolation conditions matched by P. Let $\prec_{\mathrm{lex}(i)}$, $1 \le i \le d$, be the lexicographic order*

$$x_i \succ \cdots \succ x_d \succ x_1 \succ \cdots \succ x_{i-1}.$$

Then $\ker P$ has a universal reduced Gröbner basis G satisfying the conditions of Lemma 3 if and only if

$$\mathcal{N}_{\prec_{\mathrm{lex}(1)}}(\ker\Lambda), \mathcal{N}_{\prec_{\mathrm{lex}(2)}}(\ker\Lambda), \ldots, \mathcal{N}_{\prec_{\mathrm{lex}(d)}}(\ker\Lambda)$$

are identical.

Proof. One direction of the proof is obvious due to the fact that $\ker P$ has a universal reduced Gröbner basis G satisfying the conditions of Lemma 3.

To prove the converse, assume that $G_{\prec_{\text{lex}(i)}}$, $1 \le i \le d$, is the reduced Gröbner basis for $\ker\Lambda$ w.r.t. $\prec_{\text{lex}(i)}$. Indeed, if

$$\mathcal{N}_{\prec_{\text{lex}(1)}}(\ker\Lambda) = \mathcal{N}_{\prec_{\text{lex}(2)}}(\ker\Lambda) = \cdots = \mathcal{N}_{\prec_{\text{lex}(d)}}(\ker\Lambda) = \mathcal{O},$$

then it is easy to prove

$$G_{\prec_{\text{lex}(1)}} = G_{\prec_{\text{lex}(2)}} = \cdots = G_{\prec_{\text{lex}(d)}} = G.$$

Suppose that $G = \{g_1, \ldots, g_m\}$ and $\mathcal{C}[\mathcal{O}] = \{\boldsymbol{x}^{\boldsymbol{\alpha}^{(1)}}, \ldots, \boldsymbol{x}^{\boldsymbol{\alpha}^{(m)}}\}$. Then rearranging the elements of $\mathcal{C}[\mathcal{O}]$ appropriately, we have

$$\mathrm{LT}_{\prec_{\text{lex}(1)}}(g_j) = \mathrm{LT}_{\prec_{\text{lex}(2)}}(g_j) = \cdots = \mathrm{LT}_{\prec_{\text{lex}(d)}}(g_j) = \boldsymbol{x}^{\boldsymbol{\alpha}^{(j)}}, \quad \forall 1 \le j \le m.$$

Since for arbitrary fixed $1 \le i \le d$, G is the reduced Gröbner basis for $\ker\Lambda$ w.r.t. $\prec_{\text{lex}(i)}$, it follows that the polynomials of G have the form

$$g_j = \boldsymbol{x}^{\boldsymbol{\alpha}^{(j)}} - \sum_{\substack{\boldsymbol{x}^{\boldsymbol{\beta}} \prec_{\text{lex}(i)} \boldsymbol{x}^{\boldsymbol{\alpha}^{(j)}} \\ \boldsymbol{x}^{\boldsymbol{\beta}} \in \mathcal{O}}} c_{j,\boldsymbol{\beta}} \boldsymbol{x}^{\boldsymbol{\beta}}, \quad \text{for all } 1 \le j \le m.$$

Furthermore, by the property of lexicographic order, for arbitrary $1 \le i \le d$, $\boldsymbol{x}^{\boldsymbol{\beta}} \prec_{\text{lex}(i)} \boldsymbol{x}^{\boldsymbol{\alpha}^{(j)}}$ implies that $0 \le \boldsymbol{\beta} < \boldsymbol{\alpha}^{(j)}$. Hence, we can deduce that the polynomials of G have the form:

$$g_j = \boldsymbol{x}^{\boldsymbol{\alpha}^{(j)}} - \sum_{0 \le \boldsymbol{\beta} < \boldsymbol{\alpha}^{(j)}} c_{j,\boldsymbol{\beta}} \boldsymbol{x}^{\boldsymbol{\beta}}, \quad \text{for all } 1 \le j \le m.$$

This completes the proof. □

Moreover, Proposition 7 coupled with Theorem 6 immediately implies the following useful corollary.

Corollary 8. *Let P be an ideal projector, and $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ the set of interpolation conditions matched by P. If*

$$\mathcal{N}_{\prec_{\text{lex}(1)}}(\ker\Lambda) = \mathcal{N}_{\prec_{\text{lex}(2)}}(\ker\Lambda) = \cdots = \mathcal{N}_{\prec_{\text{lex}(d)}}(\ker\Lambda) = \mathcal{O}, \tag{8}$$

where $\mathcal{N}_{\prec_{\text{lex}(i)}}(\ker\Lambda)$, $1 \le i \le d$ are as above, then the ideal projector P onto

$$V = \operatorname{span}_{\mathbb{F}}\{\boldsymbol{x}^{\boldsymbol{\beta}} : \boldsymbol{x}^{\boldsymbol{\beta}} \in \mathcal{O}\} \tag{9}$$

has a "good" error formula.

Remark 1. Let $\boldsymbol{\xi}^{(1)}, \ldots, \boldsymbol{\xi}^{(\mu)} \in \mathbb{F}^d$ be distinct points and $\mathcal{A}^{(1)}, \ldots, \mathcal{A}^{(\mu)} \subset \mathbb{N}^d$ lower sets. Suppose that the set of interpolation conditions has the form

$$\Lambda = \{\delta_{\boldsymbol{\xi}^{(k)}} \circ D^{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{A}^{(k)} \text{ for all } 1 \le k \le \mu\},$$

where $\delta_{\boldsymbol{\xi}^{(k)}}$ denotes the evaluation functional at the site $\boldsymbol{\xi}^{(k)}$, then $\mathcal{N}_{\prec_{\text{lex}(i)}}(\ker\Lambda)$ can be directly computed by the fast algorithms given in [3,8] without computing the Gröbner basis for $\ker\Lambda$.

Example 4. Suppose that the set of interpolation conditions matched by P is as follows:

$$\Lambda = \{\delta_{(0,0)}, \delta_{(0,0)} \circ D^{(0,1)}, \delta_{(0,0)} \circ D^{(1,0)}, \delta_{(0,1)}, \delta_{(0,1)} \circ D^{(1,0)}, \delta_{(1,0)}, \delta_{(1,0)} \circ D^{(1,0)}\}.$$

Since

$$\mathcal{N}_{\prec_{\mathrm{lex}(1)}}(\ker\Lambda) = \mathcal{N}_{\prec_{\mathrm{lex}(2)}}(\ker\Lambda) = \{1, x_2, x_2^2, x_1, x_1x_2, x_1^2, x_1^3\},$$

then P onto $\mathrm{span}_{\mathbb{F}}\{1, x_2, x_2^2, x_1, x_1x_2, x_1^2, x_1^3\}$ has a “good” error formula. □

As mentioned by Carl de Boor in [7], the existence of a “good” error formula for an ideal projector restricts the range of ideal projector to be of least degree. The following theorem is a particular case of this fact. Here, we also provide a simple proof, for completeness.

We denote by Π_r the subspace of polynomials in $\mathbb{F}[\boldsymbol{x}]$ of total degree at most r. Suppose $\Lambda = \{\lambda_1, \ldots, \lambda_n\} \subset (\mathbb{F}[\boldsymbol{x}])'$ and $f \in \mathbb{F}[\boldsymbol{x}]$, we write $\Lambda(f) = (\lambda_1 f, \lambda_2 f, \ldots, \lambda_n f)^T$. For a finite set $F = \{f_1, \ldots, f_k\} \subset \mathbb{F}[\boldsymbol{x}]$, $\Lambda(F)$ signifies the $n \times k$ matrix whose columns are $\Lambda(f_i)$, $1 \le i \le k$.

Theorem 9. *Let $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ be the set of interpolation conditions. If Λ satisfies condition (8), then V defined in (9) is the minimal degree interpolation space w.r.t. Λ.*

Proof. Suppose that the maximal total degree of the monomials in $\mathcal{O}$ is r, then $V \subset \Pi_r$. It is obvious that the interpolation problem of finding $p \in V$ such that

$$\lambda_i p = \lambda_i f, \quad \text{for all } 1 \le i \le n$$

is poised. According to Sauer’s definition (cf. [9]), we need to prove two properties on this special class of projectors. Firstly, the operator P onto V is *degree-reducing*, namely for each $f \in \Pi_k$ with $0 \le k \le r$, the interpolating polynomial Pf also belongs to Π_k. Secondly, the subspace $V \subset \Pi_r$ is of *minimal degree*, namely there is no subspace $V' \subset \Pi_{r-1}$ such that the above interpolation problem is poised.

Since each $f \in \Pi_k$ can be written in the form

$$f = \sum_{\substack{\boldsymbol{x}^{\boldsymbol{\beta}} \in \mathcal{O} \\ \boldsymbol{x}^{\boldsymbol{\beta}} \in \Pi_k}} c_{\boldsymbol{\beta}} \boldsymbol{x}^{\boldsymbol{\beta}} + \sum_{\substack{\boldsymbol{x}^{\boldsymbol{\beta}} \notin \mathcal{O} \\ \boldsymbol{x}^{\boldsymbol{\beta}} \in \Pi_k}} c_{\boldsymbol{\beta}} \boldsymbol{x}^{\boldsymbol{\beta}},$$

then

$$Pf = \sum_{\substack{\boldsymbol{x}^{\boldsymbol{\beta}} \in \mathcal{O} \\ \boldsymbol{x}^{\boldsymbol{\beta}} \in \Pi_k}} c_{\boldsymbol{\beta}} P(\boldsymbol{x}^{\boldsymbol{\beta}}) + \sum_{\substack{\boldsymbol{x}^{\boldsymbol{\beta}} \notin \mathcal{O} \\ \boldsymbol{x}^{\boldsymbol{\beta}} \in \Pi_k}} c_{\boldsymbol{\beta}} P(\boldsymbol{x}^{\boldsymbol{\beta}}). \tag{10}$$

Since V is the range of P, we have that for any $\boldsymbol{x}^{\boldsymbol{\beta}} \in \mathcal{O}$,

$$P(\boldsymbol{x}^{\boldsymbol{\beta}}) = \boldsymbol{x}^{\boldsymbol{\beta}}. \tag{11}$$

On the other hand, if $\boldsymbol{x}^{\boldsymbol{\beta}} \notin \mathcal{O}$, then there must exist some $\boldsymbol{x}^{\boldsymbol{\alpha}} \in \mathcal{C}[O]$ such that $\boldsymbol{x}^{\boldsymbol{\alpha}} | \boldsymbol{x}^{\boldsymbol{\beta}}$. From Corollary 8, it follows that for some $c_{\boldsymbol{\gamma}} \in \mathbb{F}$ with $0 \le \boldsymbol{\gamma} < \boldsymbol{\alpha}$,

$$\boldsymbol{x}^{\boldsymbol{\alpha}} - \sum_{0 \le \boldsymbol{\gamma} < \boldsymbol{\alpha}} c_{\boldsymbol{\gamma}} \boldsymbol{x}^{\boldsymbol{\gamma}} \in \ker P.$$

Multiplying the above equation by $\boldsymbol{x}^{\boldsymbol{\beta}-\boldsymbol{\alpha}}$, we get

$$\boldsymbol{x}^{\boldsymbol{\beta}} - \sum_{\boldsymbol{\beta}-\boldsymbol{\alpha} \le \boldsymbol{\gamma}+\boldsymbol{\beta}-\boldsymbol{\alpha} < \boldsymbol{\beta}} c_{\boldsymbol{\gamma}} \boldsymbol{x}^{\boldsymbol{\gamma}+\boldsymbol{\beta}-\boldsymbol{\alpha}} \in \ker P.$$

If $\boldsymbol{x}^{\boldsymbol{\gamma}+\boldsymbol{\beta}-\boldsymbol{\alpha}} \notin \mathcal{O}$, we repeat the above processing. Finally, we can find some $\boldsymbol{x}^{\boldsymbol{\beta}'} \in \mathcal{O}$ with $\boldsymbol{\beta}' < \boldsymbol{\beta}$, and associated coefficients $c_{\boldsymbol{\beta}'} \in \mathbb{F}$ such that

$$\boldsymbol{x}^{\boldsymbol{\beta}} - \sum_{\substack{\boldsymbol{x}^{\boldsymbol{\beta}'} \in \mathcal{O} \\ \boldsymbol{\beta}' < \boldsymbol{\beta}}} c_{\boldsymbol{\beta}'} \boldsymbol{x}^{\boldsymbol{\beta}'} \in \ker P.$$

Since $\boldsymbol{x}^{\boldsymbol{\beta}} \in \Pi_k$, it follows that

$$P(\boldsymbol{x}^{\boldsymbol{\beta}}) = \sum_{\substack{\boldsymbol{x}^{\boldsymbol{\beta}'} \in \mathcal{O} \\ \boldsymbol{\beta}' < \boldsymbol{\beta}}} c_{\boldsymbol{\beta}'} \boldsymbol{x}^{\boldsymbol{\beta}'} \in \Pi_k. \tag{12}$$

From (10)–(12), we can conclude that for arbitrary $f \in \Pi_k$ with $1 \leq k \leq r$, $Pf \in \Pi_k$.

To prove the minimal degree property, we set $\mathcal{O}' = \mathcal{O}'_1 \bigcup \mathcal{O}'_2$, where

$$\mathcal{O}'_1 := \{\boldsymbol{x}^{\boldsymbol{\beta}} \in \mathbb{T}(\boldsymbol{x}) : \boldsymbol{x}^{\boldsymbol{\beta}} \in \Pi_{r-1} \text{ and } \boldsymbol{x}^{\boldsymbol{\beta}} \in \mathcal{O}\},$$

and

$$\mathcal{O}'_2 := \{\boldsymbol{x}^{\boldsymbol{\beta}} \in \mathbb{T}(\boldsymbol{x}) : \boldsymbol{x}^{\boldsymbol{\beta}} \in \Pi_{r-1} \text{ and } \boldsymbol{x}^{\boldsymbol{\beta}} \notin \mathcal{O}\}.$$

Then we need only to prove that the matrix $\Lambda(\mathcal{O}')$ has rank less than n.

Recalling equality (12), we can easily see that for an arbitrary $\boldsymbol{x}^{\boldsymbol{\beta}} \in \mathcal{O}'_2$, $\Lambda(\boldsymbol{x}^{\boldsymbol{\beta}})$ linearly depends on the columns of $\Lambda(\mathcal{O}'_1)$. Equivalently, $\Lambda(\mathcal{O}')$ has rank less than or equal to $\#\mathcal{O}'_1$. Since at least one $\boldsymbol{x}^{\boldsymbol{\beta}} \in \mathcal{O}$ belongs to Π_r and not to Π_{r-1}, we have that the matrix $\Lambda(\mathcal{O}')$ has rank less than n. To sum up, we can say that $V = \mathrm{span}_{\mathbb{F}}\{\boldsymbol{x}^{\boldsymbol{\beta}} : \boldsymbol{x}^{\boldsymbol{\beta}} \in \mathcal{O}\}$ is the minimal degree interpolation space w.r.t. Λ. □

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